Content and the Internet, Part III

by

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The Algebra of Lexical Semantics

Synopsis

> *"An information space is an abstract space consisting of a number of states, which I will call information states, and a basic structure of difference relations between those states."*
>
> [Chalmers, The Conscious Mind, Oxford University Press 1996]

In Part II we proposed a computational semantic model of the lexicon. It turned out that our model lexicon is really a kind of *conceptual lexicon* (i.e., a set of concepts). This is good, since we prefer a language independent semantic model. Our model provides *a partial ordering of conceptual catagories*, where the granularity of our catagories is roughly at the level of the *lexeme*.

Like others before us, we start with a set-theoretic foundation; but unlike many, we focus on *structure* arising from a *partial ordering* of the set, rather than the notion of *bivalent truth conditions*. In this way, our model is unique in that it is essentially topological in nature.

With a fundamentally topological approach to modeling meaning, can we retain the algebraic structure which grounds the standard (truth conditional) models? In this paper, not only do we answer the question in the affirmative, but we show how an algebra of lexical semantics emerges necessarily from the topological model. In fact, Boolean algebra appears to be a *special case* of our algebra.

On the Meaning of the Meaning of Meaning

> *"In an important sense, ... we do not (explicitly) grasp the conditions of meaningfulness of our own utterances. Our meanings come to us through an opaque and distorted medium."*
>
> [McGinn, p.94]

We begin by recapitulating and extending some mathematical aspects of the model. We do this, not so much to indulge a sadistic streak in our character, as to provide the necessary mechanisms for motivating algebraic operations on semantic states.

Let us begin by reviewing the definition of a vector space:

> *A nonempty set **V** is said to be a vector space over a field F if **V** is an abelian*

group under an operation which we denote by +, and if for every $\alpha \in F$, $\mathbf{v} \in \mathbf{V}$ there is defined an element, written $\alpha\mathbf{v}$, in $\mathbf{V}$ subject to

1. $\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$;
2. $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$;
3. $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$;
4. $1\mathbf{v} = \mathbf{v}$;

for all $\alpha, \beta \in F$, $\mathbf{v}, \mathbf{w} \in \mathbf{V}$ (where "1" represents the unit element of F under multiplication).

In Part II we showed how a set S of lexemes could be represented as a vector space. We did this by first introducing a topology τ on S that is compatible with the sense of the lexemes, using the idea of a *directed set*, and then (given the separation axioms) showing an explicit one-to-one, continuous, open mapping _ from S to a subspace of the Hilbert coordinate space - a *de facto* vector space. Continuous and open with respect to τ, of course.

How did we express _? By the coordinate functions $\boldsymbol{g_k}$: S ⇨ $\mathbf{I}^{[0-1]}$. How did we define $\boldsymbol{g_k}$? By applying Urishon's lemma to the k^th chain of the directed set, where $\mathbf{A}$ = {S-root}, $\mathbf{B} = [n_m]$ (the closure of the minimal node of the chain), and the intermediate nodes of the chain take the role of the *seperating* sets $C(r/2^n)$, $U(r/2^n)$ [see Hocking and Young, p.57]. Of course in practice we can only *approximate* the continuous function $\boldsymbol{g_k}$ with step functions, the resolution being constrained by the chain length.

In other words, the k^{th} chain provides a natural mechanism for defining $\boldsymbol{g_k}$. Or to put it another way the k^{th} chain *identifies* $\boldsymbol{g_k}$.

As is well known, functions that are nicely behaved can form a vector space, and it so happens that open, continuous functions are very well behaved indeed. Let us consider the vector space $\mathbf{Q}$ spanned by the coordinate functions $\boldsymbol{g_k}$, where $\mathbf{q} \in \mathbf{Q}$ is of the form $\Sigma\lambda_k\boldsymbol{g_k}$, $\lambda \in$ R (the reals). We define a *norm* on $\mathbf{Q}$, something of the form $||\mathbf{q}|| = \int|\mathbf{q}|$, where it is understood that we integrate over S in some *topologically consistent* manner.

Let's pause for a moment to see where we are. We have a Banach space $\mathbf{Q}$. $\mathbf{Q}$ is a function space; it is *the* function space spanned by the functions $\boldsymbol{g_k}$. The functions $\boldsymbol{g_k}$ are defined by their corresponding chains. In fact the k^{th} chain *uniquely* identifies $\boldsymbol{g_k}$, so that $\{\boldsymbol{g_k}\}$ is more than simply a spanning set; it is a *basis* of $\mathbf{Q}$.

Having built the metric space $\mathbf{Q}$ in such a way as to entail the topology on S (via continuity), we might hope to leverage S into a metric space by exploiting $\mathbf{Q}$'s structure. An obvious strategy is to make use of the relationship between linear forms and norms.

With the two metrics (of S and $\mathbf{Q}$) demonstrated to be comensurable, we will have accomplished our goal of *quantifying the notion of near and far* in S. We proceed toward this goal.

Definition: If $\mathbf{V}$ is a vector space then its *dual space* is Hom($\mathbf{V}$, F).

Hom(**V**, F) (usually written $\mathbf{V}^*$) is the set of all vector space homomorphisms of **V** into F, also known as the space of *linear functionals*. So, in our case $\mathbf{Q}^*$ is the space of linear functionals over the function space $\mathbf{Q} = \{\mathbf{q}{:}S \Rightarrow R,\ \mathbf{q} = \Sigma\lambda_k \boldsymbol{g_k}\}$.

Now, consider that for any $s \in S$, the function ε_s associates the function $\boldsymbol{g_k}$ with an element of the real field: $\varepsilon_s(\boldsymbol{g_k}) = \boldsymbol{g_k}(s)$. A simple check shows linearity, i.e., $\varepsilon_s(\boldsymbol{g_k} + \boldsymbol{g_n}) = (\boldsymbol{g_k} + \boldsymbol{g_n})(s) = \boldsymbol{g_k}(s) + \boldsymbol{g_n}(s) = \varepsilon_s(\boldsymbol{g_k}) + \varepsilon_s(\boldsymbol{g_n})$. We leave it to the reader to verify scalar multiplication. So, what have we shown? *We have shown that every element of S corresponds to an element of the dual of **Q***.

Before continuing, we must reflect upon the following quote:

> *"If **V** is not finite-dimensional the dual of **V** is usually too large and wild to be of interest. For such vector spaces we often have additional structures, such as a **topology**, imposed and then, as the dual space, one does not generally take all of our dual but rather a **properly restricted** subspace."* [I.N. Herstein, Topics in Algebra, John Wiley & Sons, 1975; emphasis added]

Well, we have our topology; it is τ, inherited from our original characterization of S as a directed set. Let us "properly restrict" our notion of the dual of **Q** to those *linear functionals in the span of S:* $\Sigma\lambda_k s_k,\ \lambda \in R$, where it is understood that

$$(\lambda_i s_i + \lambda_j s_j)\boldsymbol{g_k} = \lambda_i s_i(\boldsymbol{g_k}) + \lambda_j s_j(\boldsymbol{g_k}).$$

For the finite dimensional case it is very easy to prove that a vector space and its dual are isomorphic. So *in practice* the dimension of the dual space of **Q** - i.e., the dimension of the space spanned by S in its new role as a *set of linear functionals* - is equal to the dimension of **Q**. By the way, what does the linear functional s "look" like? Well, s is the linear functional that maps $\boldsymbol{g_1}$ to $\boldsymbol{g_1}(s)$, $\boldsymbol{g_2}$ to $\boldsymbol{g_2}(s)$, ... $\boldsymbol{g_k}$ to $\boldsymbol{g_k}(s)$, ... In other words;

$$\textit{metrized}\ s = (\boldsymbol{g_1}(s), \boldsymbol{g_2}(s), \ldots \boldsymbol{g_k}(s), \ldots).$$

This last expression is nothing more or less than the result we established in Part II. But notice; in deriving our result this way we encountered the dual of **Q**, characterized as $\Sigma\lambda_k s_k$, $\lambda \in R$, $s \in S$. *In other words, the expression ($\lambda_i s_i + \lambda_j s_j$) now has meaning in a way that is consistent with the original topology τ defined on S*.

The last statment above is the keystone for much that is to be developed in this paper, so we state it once again for emphasis:

> ***The expression ($\lambda_i s_i + \lambda_j s_j$) has meaning in a way that is consistent with τ.***

On the Plausibility of the Norm $\|\mathbf{q}\| = \int|\mathbf{q}|$

Have we completed our program of showing "the two metrics (of S and **Q**) [to be] comensurable"? Well, no. We haven't really pinned down the notion of *metric* in S. We provide here a brief sketch of how one might approach this issue.

We gave a hint as to the general line of attack when we suggested that a *norm* on **Q** might exist; a norm defined by the notion of the integral $\int|\mathbf{q}|$ with respect to the topology τ on S. To firm up this notion, consider the following points:

- Do the elements of $\mathbf{Q} = \{\mathbf{q}: S \Rightarrow R,\ \mathbf{q} = \Sigma\lambda_k \boldsymbol{g_k}\}$ have compact support? Yes, because $\boldsymbol{g_k}$ is presumably continuous and open in some extension ***S*** of S and some refinement $\boldsymbol{\tau}$ of τ; ***S*** being some kind of *ultimate lexicon*.
- Is ε_s a positive Radon measure? Yes. Informally, we might consider any sequence of compact sets C_k where $\cap C_k = s$, *s* interior to C_k. The characteristic functions χ_{Ck} converge weakly (in the dual): $\varepsilon_s(\mathbf{q}) = \text{limit } k\uparrow\ \mathbf{q}(s)\chi_{Ck}(s)$. The linear form ε_s is often called the *Dirac measure* at the point *s*. Note that we have implicitly adopted the premise that S is locally compact.

Given a positive Radon measure μ on S, we can extend μ to the *upper integral* μ^* for positive functions on S. This leads to the definition of a semi-norm for functions on S, which in turn leads to the space $\_^1$(S, μ) (by completing **Q** with respect to the semi-norm). The norm on $\_^1$(S, μ) then reflects back (via duality) into S.

Note that if **Q** is convex, then S spans a set that sits on the convex hull of **Q**, just as we would expect that the so-called "pure" states should.

We refer the reader to the literature for a complete discussion of the subject. In particular, see Functional Analysis by R.E. Edwards, chapter four (Topological Duals of Certain Spaces: Radon Measures).

The point of all this discussion is that we can now confidently do simple *algebraic operations* on the elements of S *that are metric preserving*, namely vector addition and scalar multiplication.

On the Nature of the Elements of S

> *"What signifieth this? It is a Sign pointing the Way to the Truth that is beyond all Duality, beyond all Concept, beyond the accursed Dungeon of Yea and Nay."*
> [Robert Anton Wilson, Masks of the Illuminati]

Consider the lexemes s_i = [mother] and s_j = [father]. What is ($s_i + s_j$)? And in what sense is this sum compatible with the original topology τ?

We can answer this question by simply performing the computation. ($s_i + s_j$) is a vector which is very nearly colinear with s_n = [parent], and indeed [parent] is an element (of the dual of **Q**) that is entailed by both [mother] and [father]. One might say that s_n carries the potential to be instantiated as either s_i or s_j. Viewing the elements of S as *state vectors*, and adducing from this (and other examples), we see that vector addition can be interpreted as corresponding to a *superposition* of states. But this interpretation loses force as one considers vectors that are more distant. What is [human] + [bird]? How about [turtle] + [electorate]?

Let us propose (with appropriate misgivings) that

The sum of two state vectors corresponds to
the superposition of the states of the addends.

If state vector addition corresponds to superposition of states, the question then naturally comes to mind, "What happens when we superpose a state with itself?" By Ockham's razor, the result of such an operation should yield the same state. From this we conjecture that *if a state vector corresponding to a state is multiplied by any non-zero scalar, the resulting state vector represents the same state*. Put more succinctly,

Semantic state is entailed in the direction of the state vector.

Logic, Probability and Algebra

> "*Our world is a human world, and what is conscious and not conscious, what has sensations and what doesn't, what is qualitatively similar to what and what is dissimilar, are all dependent ultimately on our human judgments of likeness and difference*."
> [Hilary Putnam, Reason Truth and History]

The concept of meaning has long been understood to be linked intimately with *truth*. By contrast, our approach has been to find a way of quantitatively characterizing meanings (lexemes) as relatively similar or disimilar. One question must be resolved in the affirmative before our model could ever be generally accepted: *Are these two approaches to meaning mutually compatible*? In this paper we intend to show that a quantitative model of that which is essentially qualitative in nature is not just compatible with, but might be a *coherent extension of* the classical "truth conditional" approach to meaning. In developing this theme we rely extensively on the work of R.I.G. Hughes [Hughes, R.I.G, The Structure And Interpretation Of Quantum Mechanics], showing how one might coherently extend classical logic (as a Boolean algebra) into a new kind of non-classical logic: a *partial Boolean algebra*. This is the program.

Truth

> "*What is truth?*"
> [John 18:38]

Truth is usually defined in terms of satisfaction conditions. Thus, an expressed lexeme asserts its predicates, and can be evaluated to be *true* or *false* on the condition that its predicates are or are not satisfied. Traditionally, this includes so-called *counter factual* assertions. For example, "unicorn" is a valid lexeme, since there are hypothetical worlds wherein the expressed concept could be satisfied. All that is required is that the inference rules expressing necessity (logical entailment) be self consistent.

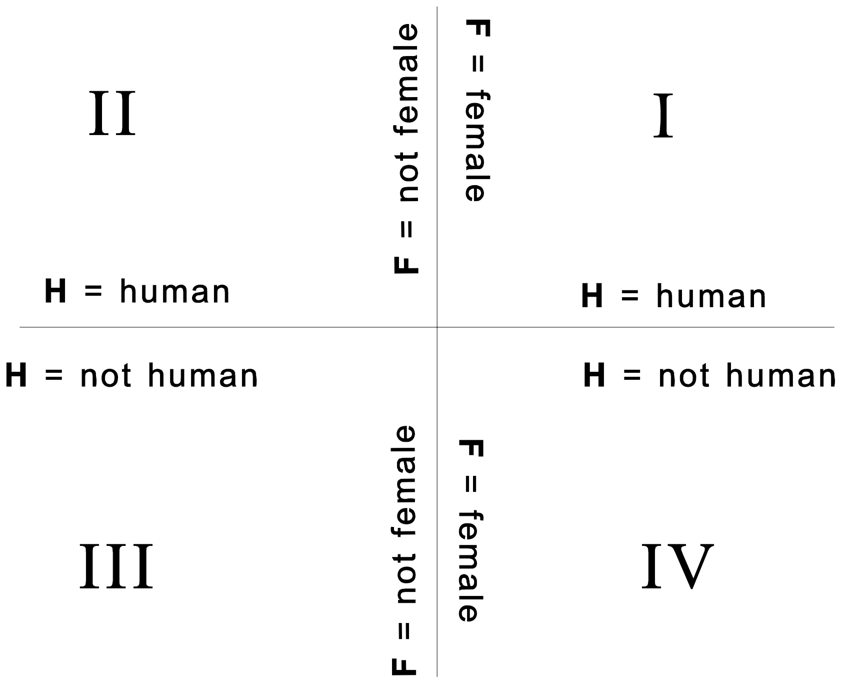
These notions are closely linked to the definition of *probability*. One typically starts with a set of *elementary events* (a.k.a. predicates), and defines a set of *events* as a subset of the power set of elementary events. So an event is very much like a lexeme. The set of events must be closed under set union, intersection and difference. A function P is introduced from the set of events to the reals, where P(union of all events) = 1. P is called a *probability function* on the events.

Suppose we could peer into the mind of a newborn baby, and examine her conceptual space. We would probably expect it to be relatively coarse grained, with respect to an adult's. Deciding to model the baby's world view, we find that her conceptual space already is amazingly complex. She has, after all, existed for nine months with an increasingly sophisticated nervous system. Just as we are about to give up the attempt, one of our collegues, Professor R.I.G., proposes a simplified model that seems to be compatible with the baby's model.

R.I.G. proposes that the baby comes minimaly hardwired to make the following distinctions (a kind of affective "case"):

- human versus not human,
- female versus not female.

The professor designates these states as **H**, **H'**, and **F**, **F'** respectively. A coarse partition of the baby's phase space is shown in Figure 1.



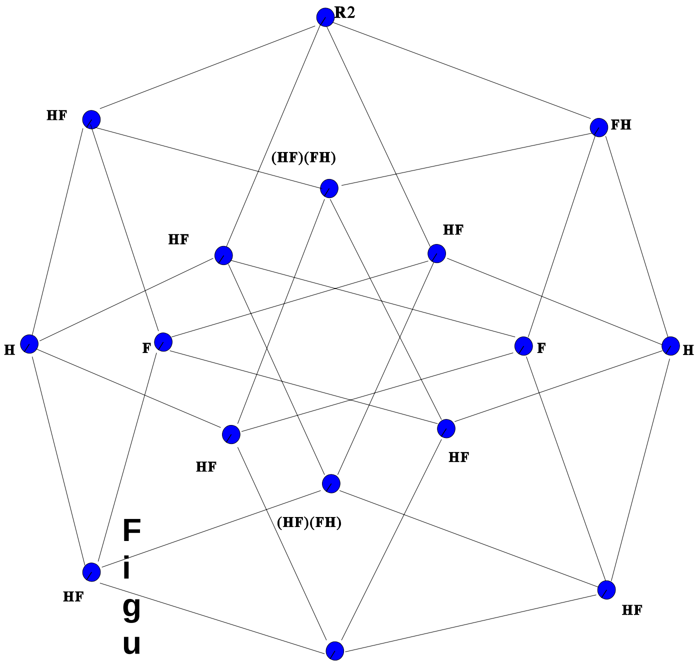


Figure 1

The baby has an "intensional stance" with respect to the states of the world. States in the first quadrant are most preferred. Human females give milk and remove irritant waste. Second quadrant states can be tolerated for a while (one *might* get waste removal service), but are less desirable. States in the fourth quadrant might correspond to a "soft" environment, womblike enough to be tolerable when not-hungry/sleeping. Quadrant three is intolerable.

In Figure 2 we model the (coarse grained) relationships among these states as a *graph*.

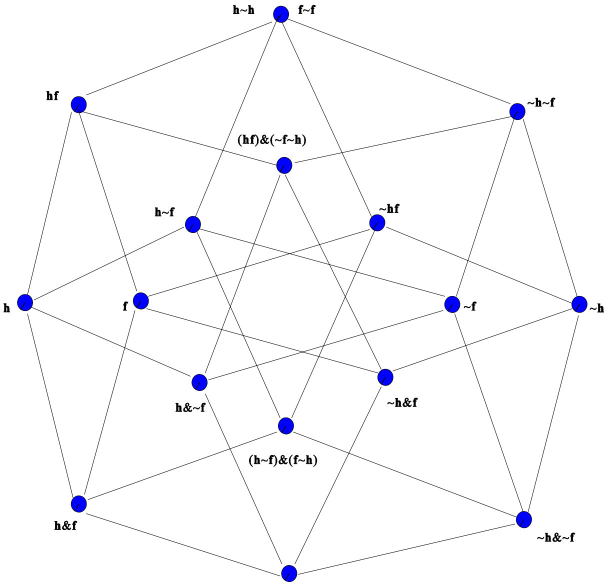


Figure 2

Clearly, we are simply modeling the inclusion relations among the principal elements of the baby's external world.

Let **h** be the assertion "the state is in **H**," and **f** be the assertion "the state is in **F**." If we use the standard logical symbols "&" for *and*, "v" for *or* and "~" for *not*, then we obtain the logical model in Figure 3.

Figure 3

Figure 3 represents the *Boolean algebra* B_{16} on the baby's phase space. The baby has only one practical mechanism to try to effect a change of state; she can cry, or she can not cry.

Boolean Algebra

> "*I suppose ... that ... whether we regard signs as the representatives of things and of their relations, or as the representatives of the conceptions and operations of the human intellect, in studying the laws of signs, we are in effect studying the manifested laws of reasoning*."
>
> [George Boole, The Laws of Thought]

A Boolean algebra B = $(B, \wedge, \vee, {}^{\perp}, 0, 1)$, where B is a set containing at least 0 and 1, $\wedge$ and $\vee$ are binary operations on B (known respectively as *meet* and *join*), and ${}^{\perp}$ is a unary operation on B such that for all a, b, c in B,

B.1 $a \vee b = b \vee a,$ $\qquad a \wedge b = b \wedge a$

B.2 $a \vee (b \vee c) = (a \vee b) \vee c,$ $\qquad a \wedge (b \wedge c) = (a \wedge b) \wedge c$

B.3 $a \vee (a \wedge b) = a,$ $\qquad a \wedge (a \vee b) = a$

B.4 $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$ $\qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

B.5 $a \vee (b \wedge b^{\perp}) = a,$ $\qquad a \wedge (b \vee b^{\perp}) = a$

We assume that the reader is familiar with this axiom system. The distributive laws (B.4) will be of particular interest to us in what follows.

We take note of one particular Boolean algebra, Z_2, where B = {0, 1} and

- $0^{\perp} = 1,$ $\qquad 1^{\perp} = 0$
- $0 \vee 1 = 1 = 1 \vee 0,$ $\qquad 0 \wedge 1 = 0 = 1 \wedge 0$
- $0 \vee 0 = 0 = 0 \wedge 0,$ $\qquad 1 \vee 1 = 1 = 1 \wedge 1.$

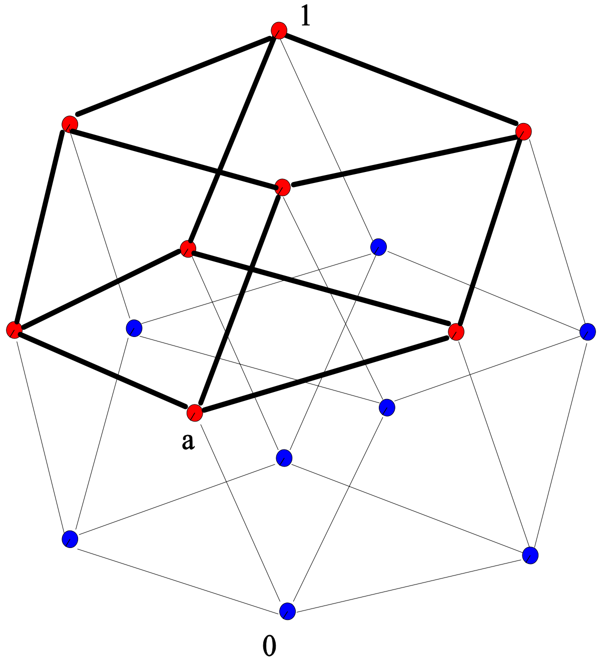
Z_2 characterizes the *law of the excluded middle*, i.e., something must be one way or another, but not both (or neither). For example, coin tossing is often modeled in Z_2; when we toss a coin we expect either heads or tails. One, and only one of the two outcomes may be realized for any given toss.

Any B is homomorphic to Z_2. That is, for any Boolean algebra we can demonstrate a mapping m:B -> Z_2 such that $m(a \vee b) = m(a) \vee m(b)$, $m(a \wedge b) = m(a) \wedge m(b)$ and $m(a^{\perp}) = (m(a))^{\perp}$ for all a, b in B, where it is understood that operations on the right side of these expressions are on Z_2. The homomorphism is said to preserve the operations $\vee$, $\wedge$ and $^{\perp}$. For B_{16}, the situation is as shown in Figure 4. Each homomorphism into Z_2 is associated with a minimal element of B_{16} which properly contains 0. The homomorphism maps a minimal element together with all its superordinating elements to 1 in Z_2. All other elements are mapped to 0.

In general, the elements of B which map to 1 under (a given) homomorphism into Z_2 form a subset. The subset is called an *Ultrafilter* on B. The Ultrafilters of B are in one-to-one correspondence with Hom(B,Z_2).

Figure 4

Note that for any Boolean algebra B, the binary relation $a \leq b$ if and only if $b = a \vee b$ (alternatively, $a = b \wedge a$) defines a *partial ordering* on B.



<u>*Lattices and Such Stuff*</u>

"*So likewise ye, except ye utter by the tongue words easy to be understood, how shall it be known what is spoken? for ye shall speak into the air.*"
[1 Corinthians, 14:9]

"*I leave it to the reader to decide whether to proceed further, or to open another beer, or both*."
[R.I.G. Hughes]

A Hilbert space H is a more general structure than is a Boolean algebra. Our task is to show that H can be viewed as a *coherent* abstraction of B.

In Part II we started with the notion of a *topology* on the set of lexemes (by defining *open* sets), and showed how a particular topology on the power set of predicates induced a *partial ordering*. We added another "constraint" by requiring that the partial ordering have a *maximum element*. Finally, we (somewhat mysteriously) "invoked" the separation axioms, which allowed us to apply Urishon's metrization theorems and thus represent the set of lexemes as a subspace of H. Note that the empty set is an element of the power set of predicates, so any partial ordering based on inclusion can be extended by appending the *minimum* element $\varnothing$.

Let *a*, *b*, *c*, *d* be elements of a partially ordered set (p.o.s.).

P.1 If $a \leq c$ and $b \leq c$, and $d \geq$ both *a* and *b* implies $c \leq d$, then *c* is called the *supremum* of (a,b), and is expressed as $a \vee b$.
P.2 Likewise the *infimum* of (a,b) is defined as a kind of greatest lower bound. The infimum of (a,b) is written as $a \wedge b$.
P.3 The *maximum element* (if any) of a p.o.s. we designate as *1*.
P.4 The *minimum element* (if any) of a p.o.s. is designated as *0*.
P.5 A p.o.s. is *complemented* if it has both a maximum and minimum element and for any element *a* there exists an element $a^{\perp}$ such that $a \vee a^{\perp} = 1$ and $a \wedge a^{\perp} = 0$; i.e., the supremum and infimum *exist* and meet the stipulated conditions.
P.6 A p.o.s. is *orthocomplemented* if it is complemented and if for any element a $(a^{\perp})^{\perp} = a$ and $a \leq b$ implies $b^{\perp} \leq a^{\perp}$.
P.7 The elements *a* and *b* of an orthocomplemented p.o.s. are said to be *orthogonal* $(a \perp b)$ if and only if $a \leq b^{\perp}$. Note that $a \perp b$ implies $b \perp a$.
P.8 A p.o.s. is *orthocomplete* if it is orthocomplemented and every pairwise orthogonal countable subset has a supremum.
P.9 A p.o.s. is orthomodular if it is orthocomplete and, if $a \leq b$ then $b = a \vee (b \wedge a^{\perp})$.

A p.o.s. is called a *lattice* if supremum and infimum (P.1 & P.2) are defined for all pairs of elements. In other words, a lattice _ is a p.o.s. for which *meet* ($\wedge$) and *join* ($\vee$) are defined for all pairs of elements of the set. All lattices satisfy axioms B.1 through B.3. Complemented lattices satisfy B.5 and the identities $1 = a \vee a^{\perp}$, $0 = a \wedge a^{\perp}$ for all *a*. Orthocomplemented lattices satisfy De Morgan's laws. A lattice which satisfies B.4 is a *distributive* lattice. An orthocomplemented distributive lattice is a Boolean algebra.

Clearly, a Boolean algebra forms a lattice _$_{B}$. To see that the set of subspaces of H, *S*(H), is a lattice _$_{S(H)}$, consider that *S*(H) is partially ordered by inclusion, and that any two subspaces $\boldsymbol{a}$ and $\boldsymbol{b}$ in *S*(H) have a minimal containing subspace $\boldsymbol{a} \vee \boldsymbol{b} = (\boldsymbol{a} \oplus \boldsymbol{b})$ and a maximal common subspace $\boldsymbol{a} \wedge \boldsymbol{b} = (\boldsymbol{a} \cap \boldsymbol{b})$ (by the completeness of H). Here we have defined *join* as the direct sum $(\boldsymbol{a} \oplus \boldsymbol{b})$, indicating the span of the two subspaces, *not their set union*.

We are now in a better position to characterize the nature of the constraints imposed by

the separation axioms
structural effect of the
reflected in the
a$\wedge$***b*** for any two
S(H), i.e., in the fact
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get us into a Hilbert
to _$_{S(\mathrm{H})}$. From a
topology we construct
must be separable, and
P.1 and P.2. T_4 is
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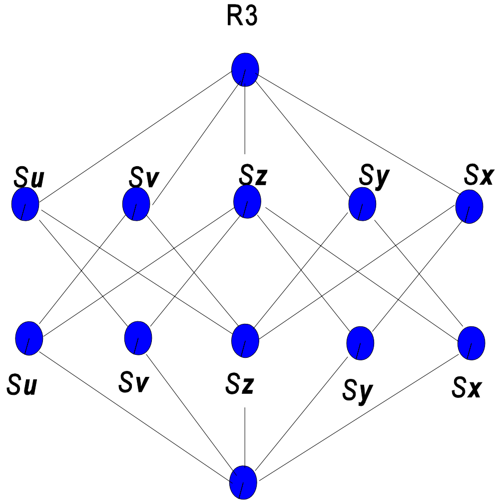

in Part II. The
separation axioms is
existence of ***a***$\vee$***b*** and
subspaces ***a*** and ***b*** in
that *S*(H) satisfies P.1
separability (countable
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separability plus the
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space H, and, *a fortiori*,
practical viewpoint, the
on the set of lexemes
structurally must satisfy
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It is a straigtforward task to show that _$_{S(\mathrm{H})}$ is an orthocomplemented lattice. In fact, _$_{S(\mathrm{H})}$ is orthomodular (P.9), but the distinction between orthomodular and orthocomplete is moot in the case where H is finite dimensional. Both _$_{B}$ and _$_{S(\mathrm{H})}$ are orthocomplete lattices; *the critical distinction is that* _$_{B}$ *is distributive, whereas* _$_{S(\mathrm{H})}$ *is not*.

To see that this is so, consider the following set of subspaces of R^3.

•	***x***	the unit vector in the *x* direction,
•	***y***	the unit vector in the *y* direction,
•	***z***	the unit vector in the *z* direction,
•	***u***	the ***x*** vector under a rotation of $\pi/4$ radians in the *x*-*y* plane,
•	***v***	the ***y*** vector under s rotation of $\pi/4$ radians in the *x*-*y* plane.

We form the sublattice G_{12} of _$_{S(R3)}$ (Figure 5), where $S_{\boldsymbol{x}}$ denotes the subspace spanned by ***x***, $S_{\boldsymbol{y}}$ the subspace spanned by ***y***, etc.

Figure 5

The operations $\vee$, $\wedge$ and $^{\perp}$ are well defined. Now consider the expression

$$S_{\boldsymbol{x}} \wedge (S_{\boldsymbol{u}} \vee S_{\boldsymbol{v}}) = S_{\boldsymbol{x}} \wedge (S_{\boldsymbol{z}}{}^{\perp}) = S_{\boldsymbol{x}} \, ,$$

whereas

$$(S_{\boldsymbol{x}} \wedge S_{\boldsymbol{u}}) \vee (S_{\boldsymbol{x}} \wedge S_{\boldsymbol{v}}) = \varnothing \vee \varnothing = \varnothing.$$

Evidently, G_{12} is not distributive.

Note that any set of mutually orthogonal "minimal" supspaces of $S(\mathrm{H})$ forms a Boolean algebra. $S(\mathrm{H})$ itself can be characterized as the union of all such algebras;

> **thus we can view $S(\mathrm{H})$ as a family of Boolean algebras pasted together to form _$_{S(\mathrm{H})}$. Such a structure is called a *partial Boolean algebra*.**

The example lattice G_{12} consists of two Boolean algebras pasted together at R^3, $S_{\boldsymbol{z}}{}^{\perp}$, $S_{\boldsymbol{z}}$, and $\varnothing$.

In fact, _$_{S(\mathrm{H})}$ satisfies another constraint known as the *coherence condition*. The coherence condition stipulates that *Boolean algebras don't overlap* in a partial Boolean algebra. Before we can develop the theme of coherence, we must introduce the notions of *compatible subspaces* and *projection operators*.

It is a strange, far path we tread toward that shadow land called meaning. Here, reader, we stop and rest for a time from our journey. Were you to leave us now, we would not blame you. It is no easy road.